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EVALUATION OF THE DISTRIBUTION FUNCTION OF THE LIMITED INFORMATION MAXIMUM LIKELIHOOD ESTIMATOR[1]

BY T. W. ANDERSON, NAOTO KUNITOMO, AND TAKAMITSU SAWA[2]

The distributions of the Limited Information Maximum Likelihood estimator for the coefficient of one endogenous variable are evaluated numerically. Tables are given for enough values of the parameters to cover all cases of interest. Comparisons are made with the Two-Stage Least Squares estimator.

1. INTRODUCTION

TO ESTIMATE THE COEFFICIENTS of a single equation in a complete system of simultaneous structural equations the Two-Stage Least Squares (TSLS) and Limited Information Maximum Likelihood (LIML) methods are commonly employed.[3] For sufficiently large sample sizes the two estimators have approximately the same distribution, but their distributions can be quite different for the sample sizes occurring in practice. The exact densities of the estimators of the coefficient of one endogenous variable have been obtained when the predetermined variables are exogenous (Richardson **[14]**, Sawa **[17]**, and Mariano and Sawa **[12]**) but their forms do not permit comparisons or analyses. Since amenable mathematical information cannot be developed, the present authors have undertaken to obtain numerical information to determine the properties of the exact cumulative distribution functions (cdf's) for a wide range of parameter values. This information makes possible the comparison of properties of the two methods. Advice can be given as to when one is preferred to the other.

Anderson and Sawa **[9]** published in this journal tables of the distribution of the TSLS estimator. In this current paper we present corresponding tables of the LIML estimator. For either estimator the distribution depends on the values of the noncentrality parameter, a standardization of the structural coefficient, and the number of exogenous variables included in the system and excluded from the relevant equation; the distribution of the LIML estimator also depends on the number of degrees of freedom in the estimator of the covariance matrix of the reduced form. The tables are given for parameter values normalized in such a

[1]This paper is a revision of Technical Report No. 319, the Economics Series, Institute for Mathematical Studies in the Social Sciences (IMSSS), Stanford University. Some details of the original paper were deleted at the suggestion of the co-editor of this journal.

[2]The authors thank the referees, the co-editor, and Yoshihiko Tsukuda for their comments on the original paper. The research was supported by National Science Foundation Grant SES 79-13976 at the Institute for Mathematical Studies in the Social Sciences, Stanford University. While the original paper was being completed, the first author was a Sherman Fairchild Distinguished Scholar at the California Institute of Technology and a Visiting Scholar at the Center for Advanced Study in the Behavioral Sciences.

[3]The method of ordinary least squares (OLS) has serious shortcomings; it yields estimators that are badly biased and inconsistent under usual conditions. However, from a mathematical viewpoint the distribution of the OLS estimator is that of the TSLS estimator with a different index.

way that with interpolation virtually all cases of interest can be analyzed. Because of the intractability of mathematical expressions, we have utilized simulation procedures, but nevertheless the tables have a high degree of accuracy, any error being in the third decimal place.

Another approach to the study of the properties of the estimators is to obtain asymptotic expansions of the distributions of the normalized estimators (Sargan and Mikhail [**16**], Anderson and Sawa [**5**], Anderson [**1**, **2**], etc.). As noted before, the leading terms are the same, but the higher-order terms are different. Anderson and Sawa [**9**] studied the accuracy of such approximations to the distributions of the TSLS estimator. The data provided in this paper permits similar evaluations for the LIML estimator.

Although the case studied here is the special one of the coefficient of one endogenous variable, it can be anticipated that the conclusions given in Section 5 will apply to some extent to cases of more coefficients. These results may suggest some properties of full information estimators.

2. THE MODEL AND ESTIMATORS

We consider the distribution of the LIML estimator of the structural parameter in an equation

$$(2.1) \qquad \boldsymbol{y}_1 = \beta \boldsymbol{y}_2 + \boldsymbol{Z}_1 \boldsymbol{\gamma}_1 + \boldsymbol{u},$$

where $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are T-component column vectors of observations on two endogenous variables, $\boldsymbol{Z}_1$ is a $T \times K_1$ matrix of observations on K_1 included exogenous variables, β is a scalar parameter, $\boldsymbol{\gamma}_1$ is a K_1-component column vector of parameters, and $\boldsymbol{u}$ is a T-component vector of unobservable disturbances. The reduced form of the system of structural equation includes

$$(2.2) \qquad (\boldsymbol{y}_1\ \boldsymbol{y}_2) = (\boldsymbol{Z}_1\, \boldsymbol{Z}_2)\begin{pmatrix} \boldsymbol{\pi}_{11} & \boldsymbol{\pi}_{12} \\ \boldsymbol{\pi}_{21} & \boldsymbol{\pi}_{22} \end{pmatrix} + (\boldsymbol{v}_1\, \boldsymbol{v}_2),$$

where $\boldsymbol{Z}_2$ is a $T \times K_2$ matrix of observations on K_2 exogenous variables in the system that are excluded from the structural equation (2.1), $\boldsymbol{\pi}_{11}$ and $\boldsymbol{\pi}_{12}$ are K_1-component vectors, $\boldsymbol{\pi}_{21}$ and $\boldsymbol{\pi}_{22}$ are K_2-component vectors of reduced form coefficients, and $(\boldsymbol{v}_1\, \boldsymbol{v}_2)$ is a $T \times 2$ matrix of disturbances.

We assume that the rows of $(\boldsymbol{v}_1\, \boldsymbol{v}_2)$ are independently normally distributed, each having mean $\boldsymbol{0}$ and nonsingular covariance matrix

$$(2.3) \qquad \boldsymbol{\Omega} = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}.$$

Since $\boldsymbol{u} = \boldsymbol{v}_1 - \beta \boldsymbol{v}_2$, the components of $\boldsymbol{u}$ are independently normally distributed with means 0 and variances $\sigma^2 = \omega_{11} - 2\beta\omega_{12} + \beta^2\omega_{22}$. The matrix $(\boldsymbol{\pi}_{21}\, \boldsymbol{\pi}_{22})$ is of rank one, $\boldsymbol{\pi}_{22}$ has at least one nonzero component, and $\boldsymbol{\pi}_{21} = \beta\boldsymbol{\pi}_{22}$. The $T \times K$ matrix $\boldsymbol{Z} = (\boldsymbol{Z}_1\, \boldsymbol{Z}_2)$ is assumed to be of rank K and $T > K$.

Let $\boldsymbol{p}_{21}$ and $\boldsymbol{p}_{22}$ be the least squares estimators for $\boldsymbol{\pi}_{21}$ and $\boldsymbol{\pi}_{22}$ and

$$A_{22.1} = Z_2'Z_2 - Z_2'Z_1(Z_1'Z_1)^{-1}Z_1'Z_2. \tag{2.4}$$

Then the LIML estimator of β is the negative of the ratio of the second to the first component of $\boldsymbol{b}$ satisfying

$$\left(\frac{1}{T}G - \lambda_1\hat{\Omega}\right)\boldsymbol{b} = \mathbf{0}, \tag{2.5}$$

where λ_1 is the smallest root of

$$\left|\frac{1}{T}G - \lambda\hat{\Omega}\right| = 0, \tag{2.6}$$

$$G = \begin{pmatrix} \boldsymbol{p}_{21}' \\ \boldsymbol{p}_{22}' \end{pmatrix} A_{22.1}(\boldsymbol{p}_{21}\ \boldsymbol{p}_{22}), \tag{2.7}$$

$$\hat{\Omega} = \frac{1}{T}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}\left[I - Z(Z'Z)^{-1}Z'\right](y_1\ y_2). \tag{2.8}$$

Replacing $\hat{\Omega}$ in (2.5) and (2.6) with Ω, we obtain the LIML estimator when the covariance matrix is Known, namely the LIMLK estimator. The LIML estimator will be denoted $\hat{\beta}_{LI}$, and the LIMLK estimator $\hat{\beta}_K$. The TSLS estimator is $\boldsymbol{p}_{21}'A_{22.1}\boldsymbol{p}_{22}/\boldsymbol{p}_{22}'A_{22.1}\boldsymbol{p}_{22}$.

We shall consider the distributions of the normalized estimators, that is

$$\frac{\sqrt{\boldsymbol{\pi}_{22}'A_{22.1}\boldsymbol{\pi}_{22}}}{\sigma}(\hat{\beta} - \beta), \tag{2.9}$$

the limiting distribution of which is $N(0,1)$ as $T \to \infty$ with $\boldsymbol{\pi}_{22}'A_{22.1}'\boldsymbol{\pi}_{22}/T$ bounded and bounded away from 0. The distribution of (2.9) for each estimator depends on K_2, the number of excluded exogenous variables, on $T - K$, the number of degrees of freedom in $\hat{\Omega}$ (for the LIML estimator), on

$$\delta^2 = \frac{\boldsymbol{\pi}_{22}'A_{22.1}\boldsymbol{\pi}_{22}}{\omega_{22}}, \tag{2.10}$$

the noncentrality parameter associated with (2.1) or alternatively $\mu^2 = (1 + \alpha^2)\delta^2$, and on

$$\alpha = \frac{\omega_{22}}{|\Omega|^{1/2}}\left(\beta - \frac{\omega_{12}}{\omega_{22}}\right), \tag{2.11}$$

the standardized structural coefficient, which measures the difference between the structural parameter (the coefficient of proportionality between the systematic parts of y_1 and y_2) and the regression coefficient of one disturbance on the other.

3. ESTIMATION OF THE CDF OF THE LIML ESTIMATOR

First, by Monte Carlo simulation we obtain the empirical cdf's of the LIML and LIMLK estimators for $\alpha = 0.0$ and $K_2 = 3$, $T - K = 10, 30$, $\delta^2 = 10, 30, 50, 100$; $K_2 = 10$, $T - K = 10, 30, 100$, $\delta^2 = 30, 50, 100, 300$; $K_2 = 30$, $T - K = 30, 100, 300$, $\delta^2 = 50, 100, 300, 1000$. For each set of values of K_2, $T - K$, and δ^2, 20,000 random matrices $\boldsymbol{G}$ and $\hat{\boldsymbol{\Omega}}$ were generated. For each pair of $\boldsymbol{G}$ and $\hat{\boldsymbol{\Omega}}$ the corresponding $\hat{\beta}_{LI}$ and $\hat{\beta}_K$ were calculated. From them one can estimate the marginal cdf's of $\hat{\beta}_{LI}$ and $\hat{\beta}_K$ for $\alpha = 0$. By making use of the transformation $\boldsymbol{Q}$ in Anderson [**1**], the LIML estimator can be expressed as a function of four χ^2's, two standard normal, and one noncentral χ^2 independent random variables. In the case of $\hat{\beta}_K$ the estimator is a function of the χ^2's, one normal, and one noncentral χ^2 independent random variables. This canonical representation of estimators facilitates efficient simulation (see Anderson, Kunitomo, and Sawa [**3**] for details).

On the basis of 20,000 replications we can calculate from the distribution of the Kolmogorov–Smirnov statistic that the empirical cdf of an estimator is within 0.01 of the true cdf everywhere with probability more than 0.99. To evaluate the accuracy of our Monte Carlo experiments, we compared the empirical and exact cdf's of the LIMLK estimator. The latter has been studied and tabulated extensively by Anderson and Sawa [**8**]. The maximum absolute value of the difference between the empirical and the exact cdf's is tabulated in Table I for the case of $\alpha = 0$.

Table I shows that the maximum value is 0.009, less than one per cent, and there is no systematic bias in our simulation. This agrees with the above theoretical consideration. On the whole, we confirm that our experiments are accurate and sampling errors are small.

The empirical cdf is an estimator of the distribution function. However, it may be possible to find a more efficient estimator of the cdf of the LIML estimator since we already know the exact distribution of the LIMLK estimator. We can make use of the fact that $\hat{\beta}_{LI}$ and $\hat{\beta}_K$ are calculated together and hence have a joint distribution with a good deal of correlation. In the simulation we observed that the association between the two estimators is high if K_2 is small, δ^2 is large,

TABLE I

MAXIMUM ABSOLUTE DIFFERENCE BETWEEN EMPIRICAL AND EXACT CDF'S OF LIMLK ESTIMATOR[a] FOR $\alpha = 0$

	Noncentrality Parameter				
	30	50	100	300	1000
$K_2 = 3$	0.7D-02	0.5D-02	0.4D-02	[b]	[b]
$K_2 = 10$	0.8D-02	0.9D-02	0.5D-02	0.7D-02	[b]
$K_2 = 30$	[b]	0.7D-02	0.5D-02	0.7D-02	0.7D-02

[a] The cdf's were calculated at values of the argument $-7.0(.5)-2.0(.2)-1.0(.1)1.0(.2)2.0(.5)7.0$.
[b] We did not perform the simulation in these cases.

and $T - K$ is large. The association decreases as K_2 increases, $T - K$ decreases, and δ^2 decreases. The effect of K_2 seems most important.

We obtain the empirical densities of the LIML and LIMLK estimators by numerical differentiation. We expect that the sampling errors of an empirical *density* in different intervals are almost independent but the values of the empirical *cdf* in neighboring intervals are highly dependent. Then in order to reduce the sampling errors of the empirical density, we use a smoothing procedure. We fitted $\phi(x/k)\sum_{i=0}^{n} c_i H_{2i}(x/k)$ to the densities by least squares, where the c_i's are coefficients, $H_{2i}(\cdot)$ is the $2i$th order Hermite polynomial and $k^2 = 1 + K_2/\mu^2$. We used only even polynomials since both the LIML and LIMLK estimators have symmetric densities when $\alpha = 0.0$. A possible justification for adjusting the scale parameter k is that k^2 is the asymptotic variance of the LIML estimator when K_2 is large (Kunitomo [**10**]). We chose $n = 3$ because use of more polynomials caused some instability in tail areas. As a result, the goodness of fit is satisfactory in each case. Let $\hat{f}_{LI}(x) = \phi(x/k)\sum_{i=0}^{n} \hat{c}_{i1} H_{2i}(x/k)$ and $\hat{f}_K(x) = \phi(x/k)\sum_{i=0}^{n} \hat{c}_{i2} H_{2i}(x/k)$ be the estimated densities of the LIML estimator and the LIMLK estimator, respectively. ($\phi(\cdot)$ and $\Phi(\cdot)$ are the density and cdf of the standard normal variable, respectively.) Then an estimate of the difference of the cdf's of these estimators is given by

$$(3.1) \qquad \hat{D}(x) = \int_0^x \left[\hat{f}_{LI}(t) - \hat{f}_K(t) \right] dt$$

$$= d_0\left[\Phi\left(\frac{x}{k} \right) - \frac{1}{2} \right] + \sum_{i=1}^{n} d_i H_{2i-1}\left(\frac{x}{k} \right)\phi\left(\frac{x}{k} \right),$$

where $d_0 = k(\hat{c}_{01} - \hat{c}_{02})$ and $d_i = k(\hat{c}_{i2} - \hat{c}_{i1})$. Then an estimate of the cdf of the LIML estimator is given:

$$(3.2) \qquad \hat{F}_{LI}(x) = F_K(x) + \hat{D}(x),$$

where $F_K(\cdot)$ is the exact distribution of the LIMLK estimator which is already known.[4] Note that $\hat{F}_{LI}(x)$ is defined for all values of x, in fact, is a function of x. This estimation procedure we used can be regarded as a control variates method in Monte Carlo experiments (e.g., see Sargan [**15**]).[5]

We can estimate the variance of $\hat{F}_{LI}(x)$ or equivalently of $\hat{D}(x)$ at any x from the covariance matrix of d_0, d_1, d_2, d_3. The covariance matrix is estimated by the program for regression of the difference of the empirical densities on $\phi(x/k)$ $H_{2i}(x/k)$, $i = 0, 1, 2, 3$. Because the variance of the difference of the empirical densities is not constant over the intervals, this estimated covariance matrix is approximate. Table II gives the estimated standard deviation of $\hat{F}_{LI}(x)$ at $x = 1$.

[4] In the technical report we give evidence that the sampling error of $\hat{F}_{LI}(x)$ is actually less than that of the empirical cdf.

[5] The authors thank a referee for pointing out some literature on this method.

TABLE II

ESTIMATED STANDARD DEVIATIONS OF CDF'S AT $x = 1.0$ FOR $\alpha = 0.0$

	δ^2	10	30	50	100
$K_2 = 3$	$T-K=10$	0.110D-02	0.892D-03	0.867D-03	0.686D-03
	$T-K=30$	0.101D-02	0.109D-02	0.737D-03	0.776D-03
	δ^2	30	50	100	300
$K_2 = 10$	$T-K=10$	0.162D-02	0.174D-02	0.111D-02	0.111D-02
	$T-K=30$	0.228D-02	0.182D-02	0.131D-02	0.138D-02
	$T-K=100$	0.115D-02	0.112D-02	0.927D-03	0.653D-03
	δ^2	50	100	300	1000
$K_2 = 30$	$T-K=30$	0.120D-02	0.154D-02	0.189D-02	0.108D-02
	$T-K=100$	0.165D-02	0.155D-02	0.157D-02	0.832D-03
	$T-K=300$	0.205D-02	0.137D-02	0.142D-02	0.114D-02

The standard deviation is 0 at $x = 0$ and approaches 0 as $x \to \infty$. The largest value in Table II is 0.00228; many of the other values are considerably smaller. Roughly speaking, we have assurance at the 99 per cent confidence level that the error in our estimate is less than 0.005; this is considerably smaller than the value of 0.01 from the Kolmogorov–Smirnov statistic for the empirical distribution. However, it should be noted that the standard deviation at some other x's can be larger than the above values in Table II.

Finally, for $0 < \alpha$ and $0 < \alpha x + \mu$, the cdf of (2.9) at x is

$$\text{(3.3)} \qquad \operatorname{pr}\left\{ \frac{\mu\hat{\beta}}{1-\alpha\hat{\beta}} \leq x \,|\, \alpha = 0 \right\} = \operatorname{pr}\left\{ \mu\hat{\beta} < \frac{x}{1 + \frac{\alpha}{\mu} x} \,\middle|\, \alpha = 0 \right\} + \operatorname{pr}\left\{ \mu\hat{\beta} > \frac{\mu}{\alpha} \,\middle|\, \alpha = 0 \right\}.$$

Then by using this formula we tabulate the estimated cdf of the LIML estimator for $\alpha = 1.0$ and 5.0. In most cases, the second term of (3.3) is found to be negligible numerically (though it does imply a small negative median bias). Note that the probabilities for $\alpha = 0$ are available for all x because we have the fitted function.

4. DISCUSSION OF THE DISTRIBUTIONS

4.1. *The Distribution of the LIML Estimator*

The distributions are tabulated in standardized terms, that is, of (2.9). The tabulation makes comparisons and interpolation easier. The table includes the three quartiles, the 2.5 and 97.5 percentiles and the interquartile range of the distribution for each case. The asymptotic standard deviation (ASD) of $\hat{\beta}$ is

$$(4.1)\qquad \frac{\sigma}{\delta\sqrt{\omega_{22}}} = \frac{\sqrt{1+\alpha^2}\sqrt{|\Omega|}}{\delta\omega_{22}} .$$

The spread of the distribution of the (unstandardized) estimator increases with $|\alpha|$ and decreases with δ. The estimators which we wish to compare LIML with, LIMLK and TSLS, have the same asymptotic standard deviation. In the remainder of the discussion we consider the normalized distributions (as tabulated).

For $\alpha = 0$ the densities are symmetric. As α increases, there is some slight asymmetry (see (3.3)), but the median is close to 0. For given α, K_2, and $T-K$ the lack of symmetry decreases as δ^2 increases. For given α, δ^2, and $T-K$ the asymmetry increases with K_2. In case of asymmetry ($\alpha > 0$) the median of $\hat{\beta}_{LI} - \beta$ is slightly negative (the median of $\hat{\beta}_{LI}$ is less than β).

The distributions have relatively long tails (in agreement with the moments not existing). As $\delta^2 \to \infty$, the distributions approach $N(0,1)$; however, for small values of δ^2 there is an appreciable probability outside of 3 or 4 ASD's. As δ^2 increases, the spread of the normalized distribution decreases. For given α, K_2, and δ^2, the spread decreases as $T-K$ increases, corresponding to $\hat{\Omega}$ being a better estimate of Ω. The spread tends to increase with K_2 and decrease with α. These observations about the spread agree with the mean and variance of the approximate distributions of (2.9) which are to orders μ^{-1} and μ^{-2}, respectively,

$$(4.2)\qquad \frac{\alpha}{\mu} = \frac{\alpha}{\delta\sqrt{1+\alpha^2}} ,$$

$$(4.3)\qquad 1 + \frac{1}{\mu^2}\left[K_2 + 2 + 8\alpha^2\right] = 1 + \frac{1}{\delta^2}\left[8 + \frac{K_2 - 6}{1+\alpha^2}\right].$$

4.2. *Comparison with the LIMLK Estimator*

The LIML and LIMLK estimators differ only in that the covariance matrix Ω is estimated to obtain the LIML estimator. The distributions of the two estimators have similar features; the LIML distribution tends to be a little more spread out than the distribution of the corresponding LIMLK estimator. Anderson, Kunitomo, and Sawa [3] gave some figures which graph the difference between the estimated cdf of the LIML estimator and the exact cdf of the LIMLK estimator. The difference of the two cdf's decreases as the noncentrality parameter δ^2 increases and as the number of degrees of freedom $T-K$ increases. As the noncentrality parameter increases, the sampling error in $\hat{\Omega}$ becomes less important and hence there is less difference between LIML based on $\hat{\Omega}$ and LIMLK based on Ω. That the difference decreases as $T-K$ increases is due to reducing the sampling error in $\hat{\Omega}$. When the number of excluded exogenous variables K_2 is as small as 3, the difference between the cdf's is less than 0.01 and can be

TABLE III

ESTIMATED DISTRIBUTION FUNCTION OF LIML ESTIMATOR

		$T-K=10, K_2=3, \alpha=0.0$				$T-K=30, K_2=3, \alpha=0.0$			
x	Normal	$\delta^2=10$	30	50	100	$\delta^2=10$	30	50	100
−3.0	0.001	0.036	0.006	0.006	0.002	0.032	0.007	0.004	0.002
−2.5	0.006	0.051	0.015	0.013	0.003	0.046	0.016	0.011	0.003
−2.0	0.023	0.076	0.038	0.032	0.027	0.070	0.038	0.031	0.026
−1.4	0.081	0.134	0.099	0.092	0.086	0.128	0.098	0.090	0.085
−1.0	0.159	0.202	0.174	0.168	0.163	0.197	0.172	0.167	0.162
−0.8	0.212	0.248	0.225	0.220	0.216	0.243	0.223	0.219	0.215
−0.6	0.274	0.302	0.284	0.281	0.278	0.298	0.283	0.279	0.277
−0.4	0.345	0.363	0.352	0.349	0.347	0.360	0.350	0.348	0.346
−0.2	0.421	0.430	0.424	0.423	0.422	0.429	0.423	0.422	0.422
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.570	0.576	0.577	0.578	0.571	0.577	0.578	0.578
0.4	0.655	0.637	0.648	0.651	0.653	0.640	0.650	0.652	0.654
0.6	0.726	0.698	0.716	0.719	0.722	0.702	0.717	0.721	0.723
0.8	0.788	0.752	0.775	0.780	0.784	0.757	0.777	0.781	0.785
1.0	0.841	0.798	0.826	0.832	0.837	0.804	0.828	0.833	0.838
1.4	0.919	0.866	0.901	0.908	0.914	0.872	0.902	0.910	0.915
2.0	0.977	0.924	0.962	0.968	0.973	0.930	0.962	0.969	0.974
2.5	0.994	0.949	0.985	0.987	0.992	0.954	0.984	0.989	0.992
3.0	0.999	0.964	0.994	0.994	0.998	0.968	0.993	0.996	0.998
X025	−1.96	−3.77	−2.24	−2.13	−2.03	−3.42	−2.25	−2.11	−2.02
L. QT	−0.67	−0.79	−0.71	−0.70	−0.69	−0.77	−0.71	−0.70	−0.68
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.79	0.71	0.70	0.69	0.77	0.71	0.70	0.68
X975	1.96	3.77	2.24	2.13	2.03	3.44	2.25	2.11	2.02
IQR	1.35	1.58	1.42	1.40	1.37	1.55	1.41	1.39	1.37

		$T-K=10, K_2=10, \alpha=0.0$				$T-K=30, K_2=10, \alpha=0.0$			
x	Normal	$\delta^2=30$	50	100	300	$\delta^2=30$	50	100	300
−3.0	0.001	0.036	0.018	0.007	0.003	0.020	0.009	0.004	0.003
−2.5	0.006	0.054	0.034	0.017	0.009	0.036	0.020	0.011	0.009
−2.0	0.023	0.086	0.064	0.040	0.028	0.065	0.044	0.033	0.026
−1.4	0.081	0.154	0.131	0.103	0.090	0.132	0.112	0.095	0.087
−1.0	0.159	0.226	0.205	0.183	0.169	0.208	0.191	0.175	0.165
−0.8	0.212	0.271	0.252	0.235	0.231	0.255	0.242	0.226	0.218
−0.6	0.274	0.322	0.307	0.293	0.281	0.309	0.300	0.286	0.279
−0.4	0.345	0.378	0.367	0.359	0.350	0.369	0.363	0.353	0.348
−0.2	0.421	0.438	0.432	0.428	0.424	0.434	0.430	0.425	0.423
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.562	0.568	0.572	0.576	0.566	0.570	0.575	0.577
0.4	0.655	0.622	0.633	0.641	0.650	0.631	0.637	0.647	0.652
0.6	0.726	0.678	0.693	0.707	0.719	0.691	0.700	0.714	0.721
0.8	0.788	0.729	0.748	0.765	0.779	0.745	0.758	0.774	0.782
1.0	0.841	0.774	0.795	0.817	0.831	0.792	0.809	0.825	0.835
1.4	0.919	0.846	0.869	0.897	0.910	0.868	0.888	0.905	0.913
2.0	0.977	0.914	0.936	0.960	0.972	0.935	0.956	0.967	0.974
2.5	0.994	0.946	0.966	0.983	0.992	0.964	0.980	0.989	0.992
3.0	0.999	0.964	0.982	0.993	0.997	0.980	0.991	0.996	0.997
X025	−1.96	−3.60	−2.74	−2.29	−2.05	−2.81	−2.35	−2.14	−2.02
L. QT	−0.67	−0.89	−0.81	−0.75	−0.70	−0.82	−0.77	−0.72	−0.69
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.89	0.81	0.75	0.70	0.82	0.77	0.72	0.69
X975	1.96	3.60	2.74	2.29	2.05	2.81	2.35	2.14	2.02
IQR	1.35	1.77	1.62	1.49	1.40	1.64	1.54	1.43	1.38

TABLE III (Continued)

x	Normal	$T-K=100$, $K_2=10$, $\alpha=0.0$ $\delta^2=30$	50	100	300	$T-K=30$, $K_2=30$, $\alpha=0.0$ $\delta^2=50$	100	300	1000
− 3.0	0.001	0.016	0.008	0.004	0.002	0.047	0.017	0.006	0.002
− 2.5	0.006	0.030	0.019	0.011	0.008	0.069	0.034	0.015	0.007
− 2.0	0.023	0.058	0.042	0.032	0.025	0.108	0.066	0.038	0.026
− 1.4	0.081	0.124	0.107	0.092	0.085	0.182	0.139	0.102	0.087
− 1.0	0.159	0.200	0.183	0.172	0.164	0.253	0.217	0.182	0.165
− 0.8	0.212	0.248	0.234	0.224	0.217	0.296	0.265	0.234	0.218
− 0.6	0.274	0.304	0.293	0.284	0.278	0.342	0.318	0.293	0.280
− 0.4	0.345	0.365	0.357	0.352	0.347	0.393	0.376	0.358	0.349
− 0.2	0.421	0.432	0.427	0.425	0.422	0.446	0.437	0.427	0.423
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.568	0.573	0.575	0.578	0.554	0.563	0.573	0.577
0.4	0.655	0.635	0.643	0.648	0.653	0.607	0.624	0.642	0.651
0.6	0.726	0.696	0.707	0.716	0.722	0.658	0.682	0.707	0.720
0.8	0.788	0.752	0.766	0.776	0.783	0.704	0.735	0.766	0.782
1.0	0.841	0.800	0.817	0.828	0.838	0.747	0.783	0.818	0.835
1.4	0.919	0.876	0.893	0.908	0.915	0.818	0.861	0.898	0.913
2.0	0.977	0.942	0.958	0.968	0.975	0.892	0.934	0.962	0.974
2.5	0.994	0.970	0.981	0.989	0.993	0.931	0.966	0.985	0.993
3.0	0.999	0.984	0.992	0.996	0.998	0.953	0.983	0.994	0.998
X025	− 1.96	− 2.64	− 2.32	− 2.14	− 2.00	− 3.92	− 2.73	− 2.23	− 2.02
L. QT	− 0.67	− 0.79	− 0.74	− 0.71	− 0.69	− 1.01	− 0.86	− 0.74	− 0.69
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.79	0.74	0.71	0.69	1.01	0.86	0.74	0.69
X975	1.96	2.64	2.32	2.14	2.00	3.92	2.73	2.23	2.02
IQR	1.35	1.59	1.48	1.42	1.37	2.03	1.72	1.48	1.39

x	Normal	$T-K=100$, $K_2=30$, $\alpha=0.0$ $\delta^2=50$	100	300	1000	$T-K=300$, $K_2=30$, $\alpha=0.0$ $\delta^2=50$	100	300	1000
− 3.0	0.001	0.025	0.007	0.002	0.002	0.020	0.006	0.002	0.002
− 2.5	0.006	0.045	0.020	0.009	0.007	0.037	0.017	0.009	0.007
− 2.0	0.023	0.082	0.050	0.029	0.026	0.072	0.044	0.029	0.025
− 1.4	0.081	0.157	0.121	0.093	0.086	0.146	0.115	0.092	0.085
− 1.0	0.159	0.231	0.199	0.173	0.163	0.221	0.193	0.171	0.163
− 0.8	0.212	0.277	0.249	0.226	0.216	0.268	0.244	0.224	0.215
− 0.6	0.274	0.327	0.305	0.287	0.277	0.321	0.300	0.285	0.277
− 0.4	0.345	0.382	0.367	0.353	0.347	0.378	0.364	0.352	0.347
− 0.2	0.421	0.440	0.432	0.425	0.422	0.438	0.430	0.424	0.422
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.560	0.568	0.575	0.578	0.562	0.570	0.576	0.578
0.4	0.655	0.618	0.633	0.647	0.653	0.622	0.636	0.648	0.653
0.6	0.726	0.673	0.695	0.713	0.723	0.679	0.700	0.715	0.723
0.8	0.788	0.723	0.751	0.774	0.784	0.732	0.756	0.776	0.785
1.0	0.841	0.769	0.801	0.827	0.837	0.779	0.807	0.829	0.837
1.4	0.919	0.843	0.879	0.907	0.914	0.854	0.885	0.908	0.915
2.0	0.977	0.918	0.950	0.971	0.974	0.928	0.956	0.971	0.975
2.5	0.994	0.955	0.980	0.991	0.993	0.963	0.983	0.991	0.993
3.0	0.999	0.975	0.993	0.998	0.998	0.980	0.994	0.998	0.998
X025	− 1.96	− 3.00	− 2.39	− 2.06	− 2.02	− 2.80	− 2.30	− 2.06	− 2.00
L. QT	− 0.67	− 0.92	− 0.80	− 0.72	− 0.69	− 0.88	− 0.78	− 0.71	− 0.68
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.92	0.80	0.72	0.69	0.88	0.78	0.71	0.68
X975	1.96	3.00	2.39	2.06	2.02	2.80	2.30	2.06	2.00
IQR	1.35	1.83	1.59	1.43	1.37	1.76	1.56	1.42	1.37

TABLE III (Continued)

x	Normal	$T-K=10, K_2=3, \alpha=1.0$ $\delta^2=10$	30	50	100	$T-K=30, K_2=3, \alpha=1.0$ $\delta^2=10$	30	50	100
−3.0	0.001	0.006	0.000	0.002	0.000	0.003	0.001	0.000	0.000
−2.5	0.006	0.008	0.000	0.003	0.002	0.005	0.002	0.001	0.002
−2.0	0.023	0.014	0.006	0.010	0.011	0.009	0.007	0.008	0.011
−1.4	0.081	0.052	0.053	0.059	0.063	0.045	0.053	0.057	0.062
−1.0	0.159	0.136	0.135	0.139	0.144	0.127	0.133	0.138	0.143
−0.8	0.212	0.199	0.195	0.198	0.201	0.188	0.193	0.197	0.200
−0.6	0.274	0.272	0.264	0.266	0.268	0.264	0.263	0.264	0.267
−0.4	0.345	0.353	0.341	0.341	0.342	0.347	0.339	0.340	0.341
−0.2	0.421	0.432	0.421	0.421	0.420	0.423	0.420	0.420	0.420
0.0	0.500	0.505	0.500	0.500	0.500	0.499	0.500	0.500	0.500
0.2	0.579	0.575	0.575	0.576	0.578	0.569	0.576	0.577	0.578
0.4	0.655	0.634	0.645	0.647	0.650	0.631	0.646	0.648	0.651
0.6	0.726	0.689	0.705	0.710	0.715	0.686	0.706	0.712	0.716
0.8	0.788	0.733	0.759	0.766	0.773	0.735	0.760	0.767	0.774
1.0	0.841	0.772	0.803	0.813	0.822	0.775	0.805	0.814	0.823
1.4	0.919	0.830	0.872	0.883	0.896	0.835	0.874	0.885	0.897
2.0	0.977	0.888	0.935	0.945	0.957	0.891	0.935	0.947	0.958
2.5	0.994	0.918	0.962	0.972	0.982	0.922	0.962	0.973	0.982
3.0	0.999	0.940	0.979	0.985	0.992	0.941	0.978	0.986	0.992
X025	−1.96	−1.71	−1.66	−1.71	−1.74	−1.60	−1.66	−1.70	−1.74
L. QT	−0.67	−0.66	−0.64	−0.64	−0.65	−0.63	−0.64	−0.64	−0.65
MEDN	0.00	−0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.88	0.76	0.74	0.72	0.87	0.76	0.73	0.71
X975	1.96	——	2.86	2.58	2.32	——	2.88	2.56	2.31
IQR	1.35	1.55	1.40	1.38	1.37	1.51	1.40	1.37	1.36

x	Normal	$T-K=10, K_2=10, \alpha=1.0$ $\delta^2=30$	50	100	300	$T-K=30, K_2=10, \alpha=1.0$ $\delta^2=30$	50	100	300
−3.0	0.001	0.012	0.007	0.000	0.000	0.003	0.000	0.000	0.000
−2.5	0.006	0.017	0.010	0.006	0.004	0.005	0.002	0.003	0.004
−2.0	0.023	0.034	0.027	0.019	0.018	0.017	0.013	0.014	0.017
−1.4	0.081	0.094	0.088	0.076	0.076	0.072	0.068	0.069	0.073
−1.0	0.159	0.177	0.169	0.159	0.157	0.157	0.154	0.151	0.153
−0.8	0.212	0.232	0.224	0.216	0.211	0.215	0.213	0.208	0.208
−0.6	0.274	0.294	0.287	0.280	0.275	0.281	0.279	0.273	0.273
−0.4	0.345	0.363	0.356	0.352	0.346	0.354	0.351	0.346	0.344
−0.2	0.421	0.431	0.428	0.425	0.423	0.427	0.426	0.422	0.422
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.565	0.569	0.572	0.576	0.569	0.571	0.575	0.577
0.4	0.655	0.625	0.634	0.641	0.649	0.633	0.638	0.646	0.651
0.6	0.726	0.678	0.692	0.703	0.715	0.690	0.699	0.710	0.717
0.8	0.788	0.726	0.742	0.759	0.774	0.740	0.751	0.767	0.777
1.0	0.841	0.766	0.786	0.807	0.825	0.783	0.798	0.815	0.828
1.4	0.919	0.831	0.853	0.881	0.900	0.851	0.870	0.889	0.903
2.0	0.977	0.893	0.916	0.945	0.963	0.915	0.948	0.952	0.964
2.5	0.994	0.923	0.945	0.972	0.986	0.945	0.965	0.978	0.987
3.0	0.999	0.940	0.964	0.984	0.995	0.964	0.981	0.990	0.995
X025	−1.96	−2.20	−2.04	−1.89	−1.87	−1.83	−1.79	−1.80	−1.84
L. QT	−0.67	−0.74	−0.71	−0.69	−0.68	−0.69	−0.68	−0.67	−0.67
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.92	0.84	0.77	0.72	0.84	0.80	0.74	0.71
X975	1.96	——	3.44	2.58	2.22	3.47	2.81	2.42	2.20
IQR	1.35	1.65	1.55	1.46	1.39	1.53	1.48	1.40	1.37

TABLE III (Continued)

x	Normal	$T-K=100, K_2=10, \alpha=1.0$ $\delta^2=30$	50	100	300	$T-K=30, K_2=30, \alpha=1.0$ $\delta^2=50$	100	300	1000
− 3.0	0.001	0.001	0.001	0.000	0.000	0.015	0.005	0.003	0.000
− 2.5	0.006	0.001	0.002	0.003	0.003	0.027	0.014	0.008	0.004
− 2.0	0.023	0.011	0.012	0.013	0.016	0.051	0.035	0.025	0.020
− 1.4	0.081	0.064	0.065	0.066	0.071	0.121	0.102	0.085	0.079
− 1.0	0.159	0.149	0.147	0.148	0.152	0.204	0.184	0.167	0.158
− 0.8	0.212	0.207	0.205	0.205	0.207	0.256	0.239	0.221	0.212
− 0.6	0.274	0.275	0.272	0.271	0.272	0.314	0.299	0.283	0.276
− 0.4	0.345	0.349	0.345	0.345	0.343	0.375	0.364	0.353	0.346
− 0.2	0.421	0.425	0.423	0.422	0.421	0.438	0.432	0.425	0.422
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.572	0.574	0.575	0.578	0.559	0.566	0.574	0.577
0.4	0.655	0.637	0.644	0.647	0.652	0.616	0.629	0.644	0.651
0.6	0.726	0.696	0.705	0.712	0.718	0.667	0.686	0.707	0.719
0.8	0.788	0.747	0.759	0.769	0.778	0.713	0.738	0.764	0.779
1.0	0.841	0.791	0.806	0.818	0.829	0.753	0.783	0.814	0.831
1.4	0.919	0.859	0.876	0.893	0.905	0.818	0.855	0.890	0.908
2.0	0.977	0.923	0.939	0.955	0.966	0.884	0.923	0.954	0.970
2.5	0.994	0.952	0.967	0.979	0.988	0.917	0.954	0.979	0.991
3.0	0.999	0.971	0.982	0.990	0.996	0.939	0.971	0.989	0.997
X025	− 1.96	− 1.75	− 1.76	− 1.78	− 1.83	− 2.57	− 2.18	− 2.00	− 1.91
L. QT	− 0.67	− 0.67	− 0.66	− 0.66	− 0.66	− 0.82	− 0.76	− 0.70	− 0.68
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.81	0.76	0.73	0.70	0.98	0.85	0.75	0.70
X975	1.96	3.14	2.72	2.39	2.16	——	3.19	2.39	2.08
IQR	1.35	1.48	1.43	1.39	1.37	1.81	1.61	1.45	1.38

x	Normal	$T-K=100, K_2=30, \alpha=1.0$ $\delta^2=50$	100	300	1000	$T-K=300, K_2=30, \alpha=1.0$ $\delta^2=50$	100	300	1000
− 3.0	0.001	0.003	0.000	0.000	0.001	0.000	0.000	0.000	0.001
− 2.5	0.006	0.008	0.004	0.002	0.004	0.003	0.002	0.002	0.004
− 2.0	0.023	0.027	0.021	0.017	0.020	0.019	0.016	0.017	0.019
− 1.4	0.081	0.096	0.084	0.075	0.078	0.085	0.077	0.075	0.077
− 1.0	0.159	0.180	0.167	0.158	0.156	0.170	0.160	0.156	0.156
− 0.8	0.212	0.235	0.222	0.213	0.210	0.227	0.217	0.211	0.209
− 0.6	0.274	0.298	0.285	0.277	0.273	0.231	0.280	0.275	0.273
− 0.4	0.345	0.364	0.354	0.348	0.344	0.359	0.351	0.347	0.344
− 0.2	0.421	0.432	0.427	0.423	0.421	0.430	0.425	0.422	0.421
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.565	0.571	0.576	0.578	0.567	0.573	0.577	0.578
0.4	0.655	0.626	0.638	0.648	0.653	0.630	0.641	0.649	0.653
0.6	0.726	0.681	0.699	0.713	0.722	0.687	0.703	0.715	0.722
0.8	0.788	0.731	0.753	0.772	0.781	0.739	0.758	0.774	0.782
1.0	0.841	0.774	0.800	0.823	0.833	0.783	0.806	0.825	0.833
1.4	0.919	0.842	0.873	0.900	0.909	0.852	0.879	0.901	0.910
2.0	0.977	0.909	0.940	0.964	0.970	0.920	0.946	0.963	0.971
2.5	0.994	0.943	0.969	0.986	0.990	0.953	0.974	0.986	0.991
3.0	0.999	0.963	0.984	0.995	0.997	0.972	0.988	0.995	0.997
X025	− 1.96	− 2.03	− 1.93	− 1.85	− 1.90	− 1.91	− 1.86	− 1.85	− 1.89
L. QT	− 0.67	− 0.75	− 0.71	− 0.68	− 0.67	− 0.75	− 0.69	− 0.68	− 0.67
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.89	0.79	0.72	0.69	0.85	0.77	0.71	0.69
X975	1.96	3.52	2.66	2.19	2.09	3.12	2.53	2.22	2.07
IQR	1.35	1.64	1.50	1.40	1.36	1.60	1.46	1.39	1.36

TABLE III (Continued)

x	Normal	$T-K=10, K_2=3, \alpha=5.0$ $\delta^2=10$	30	50	100	$T-K=30, K_2=3, \alpha=5.0$ $\delta^2=10$	30	50	100
− 3.0	0.001	0.001	0.000	0.002	0.000	0.001	0.001	0.000	0.000
− 2.5	0.006	0.001	0.000	0.002	0.000	0.001	0.001	0.000	0.000
− 2.0	0.023	0.001	0.001	0.005	0.006	0.001	0.002	0.003	0.006
− 1.4	0.081	0.007	0.033	0.044	0.053	0.008	0.033	0.042	0.052
− 1.0	0.159	0.077	0.115	0.126	0.135	0.079	0.113	0.124	0.134
− 0.8	0.212	0.149	0.179	0.186	0.194	0.151	0.177	0.185	0.193
− 0.6	0.274	0.233	0.253	0.259	0.263	0.236	0.252	0.257	0.262
− 0.4	0.345	0.326	0.336	0.337	0.340	0.332	0.334	0.336	0.339
− 0.2	0.421	0.420	0.419	0.420	0.419	0.423	0.418	0.419	0.419
0.0	0.500	0.507	0.500	0.500	0.500	0.507	0.500	0.500	0.500
0.2	0.579	0.577	0.575	0.576	0.578	0.580	0.576	0.577	0.578
0.4	0.655	0.639	0.644	0.647	0.649	0.639	0.645	0.648	0.650
0.6	0.726	0.694	0.704	0.708	0.713	0.696	0.705	0.710	0.714
0.8	0.788	0.739	0.756	0.762	0.770	0.749	0.757	0.763	0.771
1.0	0.841	0.776	0.798	0.808	0.818	0.777	0.800	0.809	0.819
1.4	0.919	0.832	0.865	0.876	0.890	0.835	0.867	0.878	0.891
2.0	0.977	0.890	0.926	0.938	0.952	0.894	0.927	0.940	0.953
2.5	0.994	0.919	0.955	0.965	0.978	0.922	0.955	0.967	0.978
3.0	0.999	0.939	0.972	0.981	0.990	0.942	0.972	0.982	0.990
X025	− 1.96	− 1.24	− 1.47	− 1.57	− 1.66	− 1.24	− 1.47	− 1.55	− 1.65
L. QT	− 0.67	− 0.56	− 0.61	− 0.62	− 0.64	− 0.57	− 0.60	− 0.62	− 0.63
MEDN	0.00	− 0.02	0.00	0.00	0.00	− 0.02	0.00	0.00	0.00
U. QT	0.67	0.86	0.77	0.75	0.72	0.81	0.77	0.75	0.72
X975	1.96	——	3.13	2.78	2.42	——	3.13	2.73	2.42
IQR	1.35	1.42	1.38	1.38	1.36	1.38	1.38	1.37	1.35

x	Normal	$T-K=10, K_2=10, \alpha=5.0$ $\delta^2=30$	50	100	300	$T-K=30, K_2=10, \alpha=5.0$ $\delta^2=30$	50	100	300
− 3.0	0.001	0.012	0.007	0.000	0.001	0.003	0.000	0.000	0.000
− 2.5	0.006	0.013	0.007	0.002	0.003	0.003	0.000	0.000	0.003
− 2.0	0.023	0.021	0.017	0.014	0.014	0.006	0.005	0.008	0.013
− 1.4	0.081	0.064	0.068	0.063	0.069	0.042	0.048	0.056	0.066
− 1.0	0.159	0.145	0.149	0.146	0.151	0.124	0.133	0.138	0.147
− 0.8	0.212	0.205	0.206	0.205	0.206	0.187	0.194	0.197	0.203
− 0.6	0.274	0.274	0.273	0.272	0.271	0.260	0.265	0.265	0.269
− 0.4	0.345	0.349	0.348	0.347	0.344	0.340	0.343	0.341	0.342
− 0.2	0.421	0.425	0.424	0.423	0.422	0.421	0.422	0.420	0.421
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.569	0.572	0.574	0.576	0.573	0.574	0.577	0.577
0.4	0.655	0.632	0.637	0.643	0.649	0.640	0.641	0.648	0.651
0.6	0.726	0.687	0.696	0.704	0.715	0.698	0.703	0.711	0.717
0.8	0.788	0.735	0.745	0.759	0.773	0.749	0.755	0.767	0.770
1.0	0.841	0.774	0.788	0.806	0.823	0.790	0.800	0.814	0.826
1.4	0.919	0.837	0.854	0.878	0.898	0.856	0.871	0.886	0.901
2.0	0.977	0.895	0.913	0.942	0.960	0.917	0.933	0.949	0.962
2.5	0.994	0.924	0.943	0.968	0.984	0.946	0.963	0.975	0.985
3.0	0.999	0.942	0.960	0.983	0.994	0.964	0.978	0.988	0.994
X025	− 1.96	− 1.89	− 1.83	− 1.77	− 1.81	− 1.56	− 1.59	− 1.67	− 1.79
L. QT	− 0.67	− 0.67	− 0.67	− 0.66	− 0.66	− 0.63	− 0.64	− 0.64	− 0.65
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.87	0.82	0.77	0.72	0.80	0.78	0.74	0.71
X975	1.96	——	3.80	2.70	2.27	3.51	2.06	2.50	2.23
IQR	1.35	1.54	1.49	1.43	1.38	1.43	1.42	1.38	1.37

TABLE III (Continued)

x	Normal	$T-K=100, K_2=10, \alpha=5.0$ $\delta^2=30$	50	100	300	$T-K=30, K_2=30, \alpha=5.0$ $\delta^2=50$	100	300	1000
− 3.0	0.001	0.001	0.001	0.000	0.000	0.011	0.003	0.002	0.000
− 2.5	0.006	0.000	0.001	0.001	0.002	0.019	0.009	0.006	0.003
− 2.0	0.023	0.002	0.004	0.008	0.012	0.034	0.024	0.020	0.018
− 1.4	0.081	0.035	0.045	0.053	0.064	0.084	0.079	0.076	0.075
− 1.0	0.159	0.116	0.126	0.134	0.146	0.164	0.162	0.157	0.154
− 0.8	0.212	0.179	0.186	0.194	0.202	0.220	0.218	0.212	0.209
− 0.6	0.274	0.254	0.258	0.263	0.268	0.285	0.283	0.277	0.274
− 0.4	0.345	0.335	0.337	0.340	0.341	0.356	0.353	0.348	0.345
− 0.2	0.421	0.419	0.419	0.420	0.420	0.428	0.427	0.423	0.421
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.576	0.577	0.577	0.578	0.567	0.570	0.575	0.578
0.4	0.655	0.644	0.647	0.649	0.652	0.630	0.636	0.646	0.652
0.6	0.726	0.704	0.709	0.713	0.718	0.685	0.695	0.710	0.719
0.8	0.788	0.755	0.762	0.769	0.777	0.734	0.748	0.768	0.779
1.0	0.841	0.798	0.808	0.817	0.827	0.774	0.792	0.817	0.831
1.4	0.919	0.864	0.877	0.890	0.902	0.839	0.862	0.890	0.907
2.0	0.977	0.925	0.938	0.952	0.964	0.898	0.926	0.954	0.968
2.5	0.994	0.954	0.966	0.977	0.986	0.926	0.954	0.978	0.990
3.0	0.999	0.971	0.980	0.990	0.995	0.944	0.970	0.989	0.997
X025	− 1.96	− 1.49	− 1.57	− 1.65	− 1.77	− 2.26	− 1.98	− 1.91	− 1.86
L. QT	− 0.67	− 0.61	− 0.62	− 0.64	− 0.65	− 0.71	− 0.70	− 0.67	− 0.67
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.78	0.75	0.73	0.71	0.87	0.81	0.74	0.70
X975	1.96	3.18	2.77	2.45	2.20	——	3.25	2.41	2.12
IQR	1.35	1.39	1.37	1.37	1.36	1.58	1.51	1.41	1.37

x	Normal	$T-K=100, K_2=30, \alpha=5.0$ $\delta^2=50$	100	300	1000	$T-K=300, K_2=30, \alpha=5.0$ $\delta^2=50$	100	300	1000
− 3.0	0.001	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000
− 2.5	0.006	0.003	0.001	0.001	0.003	0.000	0.000	0.001	0.003
− 2.0	0.023	0.011	0.011	0.012	0.018	0.004	0.007	0.012	0.017
− 1.4	0.081	0.058	0.062	0.066	0.074	0.047	0.055	0.066	0.073
− 1.0	0.159	0.141	0.144	0.148	0.152	0.130	0.137	0.146	0.152
− 0.8	0.212	0.200	0.202	0.204	0.207	0.191	0.196	0.202	0.206
− 0.6	0.274	0.269	0.269	0.271	0.271	0.262	0.265	0.269	0.271
− 0.4	0.345	0.345	0.343	0.343	0.343	0.340	0.340	0.342	0.343
− 0.2	0.421	0.422	0.422	0.421	0.420	0.420	0.420	0.420	0.420
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.579	0.573	0.575	0.577	0.579	0.575	0.577	0.578	0.579
0.4	0.655	0.640	0.645	0.650	0.654	0.644	0.648	0.651	0.654
0.6	0.726	0.699	0.708	0.716	0.722	0.706	0.712	0.718	0.722
0.8	0.788	0.751	0.763	0.775	0.781	0.759	0.768	0.777	0.782
1.0	0.841	0.795	0.809	0.825	0.833	0.803	0.815	0.827	0.833
1.4	0.919	0.862	0.880	0.900	0.908	0.873	0.886	0.901	0.909
2.0	0.977	0.923	0.944	0.963	0.969	0.934	0.950	0.963	0.970
2.5	0.994	0.952	0.970	0.986	0.989	0.963	0.976	0.986	0.990
3.0	0.999	0.969	0.984	0.996	0.997	0.979	0.989	0.995	0.997
X025	− 1.96	− 1.72	− 1.73	− 1.77	− 1.86	− 1.60	− 1.68	− 1.77	− 1.84
L. QT	− 0.67	− 0.65	− 0.65	− 0.66	− 0.66	− 0.63	− 0.64	− 0.65	− 0.66
MEDN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
U. QT	0.67	0.80	0.75	0.71	0.69	0.76	0.73	0.71	0.69
X975	1.96	3.29	2.65	2.21	2.11	2.84	2.47	2.21	2.09
IQR	1.35	1.45	1.40	1.37	1.36	1.40	1.37	1.36	1.35

ignored for practical purposes such as testing hypotheses. As K_2 increases, the maximum difference increases (for given δ^2 and $T-K$). When K_2 is as large as 30, as it can be in contemporary econometric models, the cdf of the LIML estimator may differ substantially from that of the LIMLK estimator. However, when $T-K$ is as large as 300, this difference becomes negligible.

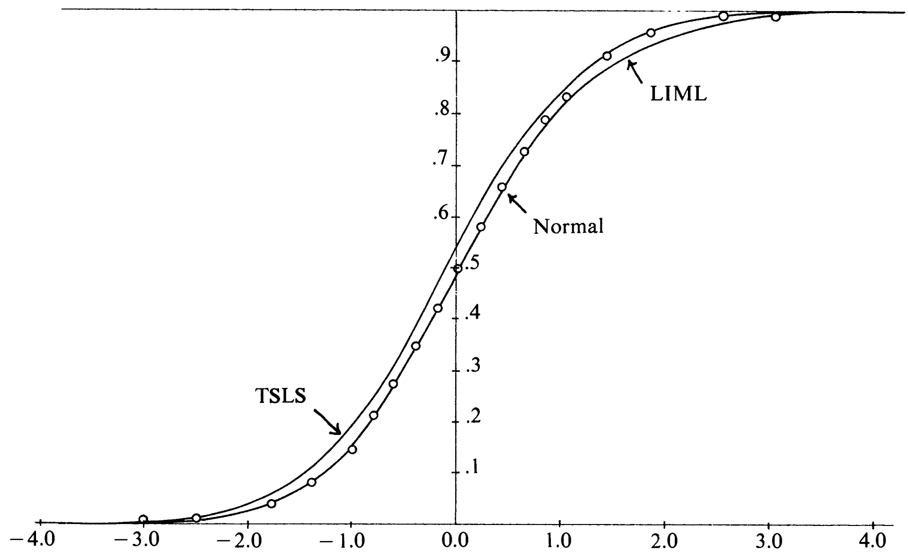
4.3. *Comparison with the TSLS Estimator*

Anderson and Sawa [9] have given tables of the distributions of the TSLS estimator and discussed their properties. These properties agree with the asymptotic expansions of the distributions (Anderson and Sawa [5]). The mean and variance of the asymptotic expansion (to 3 terms) of the TSLS estimator are

$$-\frac{(K_2-2)\alpha}{\mu}=-\frac{(K_2-2)\alpha}{\delta\sqrt{1+\alpha^2}}, \tag{4.4}$$

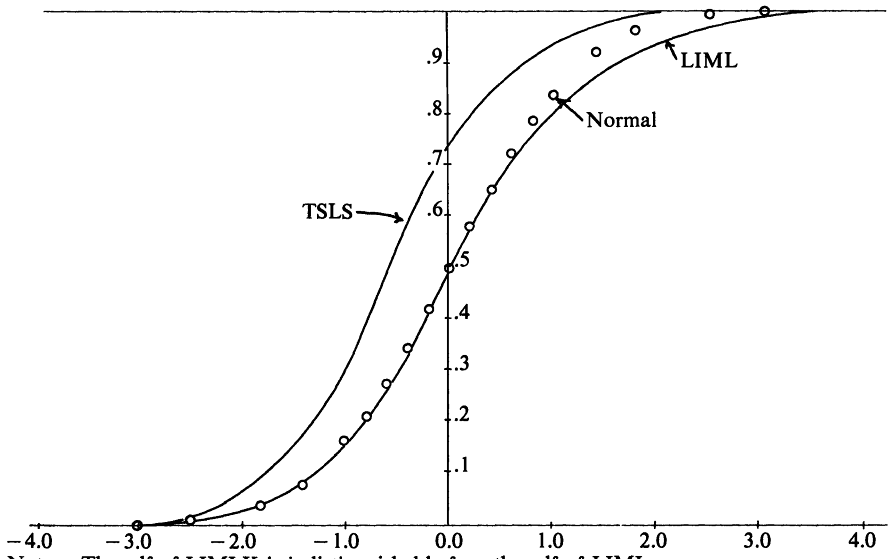
$$1-\frac{K_2-4+4(K_2-3)\alpha^2}{\mu^2}=1-\frac{4(K_2-3)-\left(3K_2-8/1+\alpha^2\right)}{\delta^2}. \tag{4.5}$$

The most striking feature is that the distribution of the TSLS estimator is skewed towards the left for $\alpha>0$ (and towards the right for $\alpha<0$), and the distortion increases with α and K_2. Figures 1, 2, and 3 show the estimated cdf's of the LIML estimator, and the exact cdf's of the LIMLK and TSLS estimators for the case when $\alpha=1.0$, $\delta^2=100.0$, and $K_2=3,10,30$. The LIML and LIMLK



Note: The cdf of LIMLK is indistinguishable from the cdf of LIML.

FIGURE 1—Cumulative distribution functions, $T-K=10$, $K_2=3$, $\alpha=1.0$, $\delta^2=100.0$.



Note: The cdf of LIMLK is indistinguishable fron the cdf of LIML.

FIGURE 2—Cumulative distribution functions, $T - K = 10$, $K_2 = 10$, $\alpha = 1.0$, $\delta^2 = 100.0$.

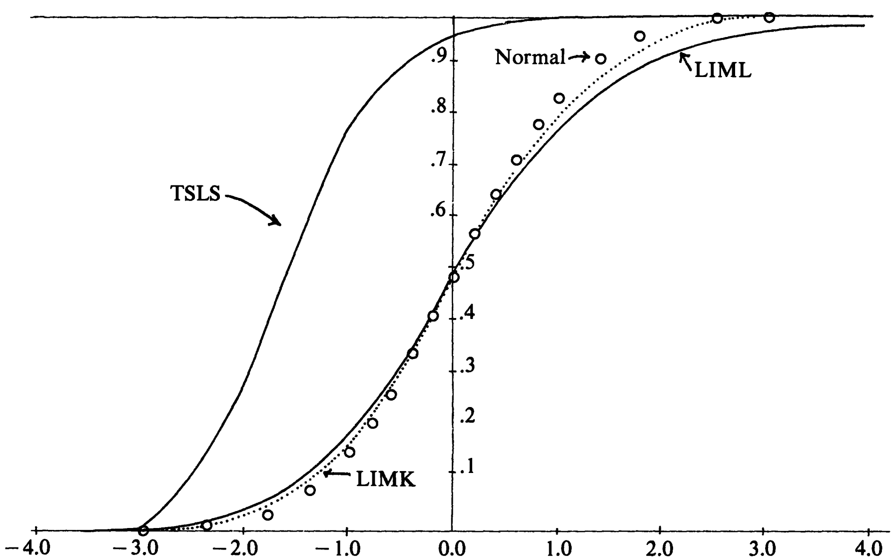


FIGURE 3—Cumulative distribution functions, $T - K = 30$, $K_2 = 30$, $\alpha = 1.0$, $\delta^2 = 100.0$.

estimators are essentially median-unbiased in each case. On the other hand the TSLS estimator is biased. As K_2 increases, this bias becomes more serious; for $K_2 = 30$ the median is about -1.6 ASD's. If K_2 is large, the TSLS estimator substantially underestimates the parameter. This fact definitely favors the LIML estimator over the TSLS estimator. However, when K_2 is as small as 3, the TSLS estimator is very similar to the LIML estimator. (When $K_2 = 1$, the TSLS, LIMLK, and LIML estimators are identical.)

The distributions of the LIML and LIMLK estimators are a little more spread out than the distributions of the TSLS estimator. This reflects the fact that the LIML and LIMLK estimators do not have moments of positive integer order. The difference between the asymptotic variance of the LIML and LIMLK estimators and that of the TSLS estimator is $2(K_2-1)(1+2\alpha^2)/\mu^2 = 2(K_2-1)[2-1/(1+\alpha^2)]/\delta^2$. The difference increases with α and K_2 and decreases with δ^2. The interquartile ranges of the LIML estimator are larger than those of the TSLS estimator in most cases.

The distributions of the LIML and LIMLK estimators approach normality faster than the distribution of the TSLS estimator, due primarily to the bias of the latter. The actual 97.5 percentiles of the TSLS estimator are substantially different from 1.96 of the standard normal when $\alpha \neq 0$ and $K_2 = 10, 30$. This implies that the conventional hypothesis testing about a structural coefficient based on the normal approximation is very likely to seriously underestimate the actual significance. The 2.5 and 97.5 percentiles of the LIML estimator are much closer to those of the standard normal distribution.

4.4. *Errors of Approximations*

Anderson and Sawa [**7**] gave a small table of the maximum difference between the cdf of the LIMLK estimator and the cdf of the normal distribution, and the asymptotic expansions to order δ^{-4}. The differences between the cdf of the LIML estimator and the approximations can be expected to be similar. In fact, we can examine the accuracy of the approximate distributions of the LIML estimator based on the asymptotic expansion by simply comparing the cdf's of the LIML and LIMLK estimators and the tables given by Anderson and Sawa [**6** and **8**].

4.5. *The Noncentrality Parameter*

The noncentrality parameter $\delta^2 = \boldsymbol{\pi}_{22}' \boldsymbol{A}_{22.1} \boldsymbol{\pi}_{22}/\omega_{22}$ plays a key role. The assumption that $\boldsymbol{\pi}_{22} \neq \mathbf{0}$ is necessary and sufficient for identification of the equation. Since we assume $\boldsymbol{A}_{22.1}$ is positive definite ($\boldsymbol{Z}$ of rank K) and $\omega_{22} > 0$ ($\boldsymbol{\Omega}$ positive definite), the condition $\delta^2 > 0$ is an equivalent condition. Each of the components of $\boldsymbol{\pi}_{22}$, $\boldsymbol{A}_{22.1}$, and ω_{22} depends on the units of measurement, but δ^2 does not. In a sense, δ^2 determines how well the equation is identified.

Given $\boldsymbol{\pi}_{22}$ and ω_{22}, the larger $\boldsymbol{A}_{22.1}$ is, the larger δ^2 is. In multivariate analysis of variance terms $\boldsymbol{\pi}_{22}'\boldsymbol{A}_{22.1}\boldsymbol{\pi}_{22}$ is the effect sum of squares due to excluded exogenous variables. In many cases it would be reasonable to expect the elements of $\boldsymbol{A}_{22.1}$ to be proportional to T, but the factors of proportionality depend on the excluded exogenous variables and their relationship to the included exogenous variables. Given the variables to be measured, the value of δ^2 and of $T-K$ would increase with T; the investigators should use as many observations as possible. One would also expect δ^2 to increase with K_2, but perhaps not proportionally (some authors assume $(1/T)\boldsymbol{A}_{22.1}$ approaches a limit in order to prove asymptotic normality of the estimators, but that is not a necessary assumption).

4.6. *Effect of Normality*

The distributions are based on the assumption that the disturbances are normally distributed; we have not investigated the distributions of the estimators in the case of nonnormality. The estimators of the coefficients of the reduced form tend to be normally distributed (by the central limit theorem). The fact that the distributions of the LIML estimator do not depend much on $T-K$ suggests that the exact distribution of the covariance matrix is not crucial to the behavior of the estimators. It can be conjectured that the comparisons of distributions are approximately valid if the distributions of the disturbances are not too far from normal.

5. CONCLUSIONS

First, the distributions of the LIML and TSLS estimators are substantially different. The LIML estimator is to be strongly preferred to the TSLS estimator, particularly if K_2 is large. In actual large econometric models, it is a common feature that K_2 is fairly large and hence the LIML method is recommended. (Kunitomo [**10**], Morimune and Kunitomo [**13**], and Kunitomo [**11**] have shown that if $K_2 \to \infty$ as $T \to \infty$ the LIML estimator is asymptotically efficient while the TSLS estimator is not consistent.)

Second, the large-sample normal approximation is fairly accurate for the LIML estimator except for small values of δ^2. Hence, the usual methods with asymptotic standard deviations give reasonable inference. On the other hand for the TSLS estimator the value of δ^2 must be very large to justify the use of procedures based on normality. These facts throw into doubt the usual justification of the TSLS estimator as asymptotically equivalent to the LIML estimator because the usual values of δ^2 are too small.

Third, in practice K_2 and $T-K$ are known, but α and δ^2 would have to be estimated. (δ^2 can be estimated by T times the larger root of (2.6).) In terms of the estimated α and δ^2 the properties of the LIML and TSLS estimators could be approximated and the investigators could choose between them. Of course, if K_2

is fairly large the TSLS estimator is very risky in the sense that it will be very biased if the (unknown) α is different from 0. Anderson, Morimune, and Sawa [4] have estimated the key parameters in a number of classical econometric studies. They find a "typical" value of α of about 1 and of $\delta^2/(TK_2)$ of about 1. In current studies with larger K_2, we expect $\delta^2/(TK_2)$ to be smaller (because of "multicollinearity").

To summarize, the most important conclusion from the study of the LIML and TSLS estimators is that the TSLS estimator can be badly biased and in that sense its use is risky. The LIML estimator, on the other hand, has a little more variability with a slight chance of extreme values, but its distribution is centered at the parameter value.

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REFERENCES

[1] ANDERSON, T. W.: "An Asymptotic Expansion of the Distribution of the Limited Information Maximum Likelihood Estimate of a Coefficient in a Simultaneous Equation System," *Journal of the American Statistical Association*, 69(1974), 565–573.

[2] ———: "Asymptotic Expansions of the Distributions of Estimates in Simultaneous Equations for Alternative Parameter Sequences," *Econometrica*, 45(1977), 509–518.

[3] ANDERSON, T. W., NAOTO KUNITOMO, AND TAKAMITSU SAWA: "Evaluation of the Distribution Function of the Limited Information Maximum Likelihood Estimator," Technical Report No. 319, the Economics Series, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1980.

[4] ANDERSON, T. W., KIMIO MORIMUNE, AND TAKAMITSU SAWA: "The Numerical Values of Some Key Parameters in Econometric Models," Technical Report No. 270, the Economics Series, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1978.

[5] ANDERSON, T. W., AND TAKAMITSU SAWA: "Distributions of Estimates of Coefficients of a Single Equation in a Simultaneous System and Their Asymptotic Expansions," *Econometrica*, 41(1973), 683–714.

[6] ———: "Distribution of a Maximum Likelihood Estimate of a Slope Coefficient: The LIML Estimate for Known Covariance Matrix," Technical Report No. 174, the Economics Series, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1975.

[7] ———: "Tables of the Distribution of the Maximum Likelihood Estimate of the Slope Coefficient and Approximations," Technical Report No. 234, the Economics Series, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1977.

[8] ———: "Table of the Exact Distribution Function of the Limited Information Maximum Likelihood Estimator When the Covariance Is Known," Technical Report No. 275, the Economics Series, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1978.

[9] ———: "Evaluation of the Distribution Function of the Two-Stage Least Squares Estimate," *Econometrica*, 47(1979), 163–182.

[10] KUNITOMO, NAOTO: "Asymptotic Expansions of the Distributions of Estimators in a Linear Functional Relationship and Simultaneous Equations," *Journal of the American Statistical Association*, 75(1980), 693–700.

[11] ———: "Asymptotic Efficiency and Higher Order Efficiency of the Limited Information Maximum Likelihood Estimator in Large Econometric Models," Technical Report No. 365, the Economics Series, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1982.

[12] MARIANO, ROBERTO S., AND TAKAMITSU SAWA: "The Exact Finite-Sample Distribution of the Limited-Information Maximum Likelihood Estimator in the Case of Two Included Endogenous Variables," *Journal of the American Statistical Association*, 67(1972), 159–163.

[13] MORIMUNE, KIMIO, AND NAOTO KUNITOMO: "Improving the Maximum Likelihood Estimate in Linear Functional Relationships for Alternative Parameter Sequences," *Journal of the American Statistical Association*, 75(1980), 230–237.

[14] RICHARDSON, DAVID H: "The Exact Distribution of a Structural Coefficient Estimator," *Journal of the American Statistical Association*, 63(1968), 1224–1226.

[15] SARGAN, J. D.: "Econometric Estimators and the Edgeworth Approximation," *Econometrica*, 44(1976), 421–448.

[16] SARGAN, J. D., AND W. J. MIKHAIL: "A General Approximation to the Distribution of Instrumental Variables Estimates," *Econometrica*, 39(1971), 131–169.

[17] SAWA, TAKAMITSU: "The Exact Sampling Distribution of Ordinary Least Squares and Two-Stage Least Squares Estimator," *Journal of American Statistical Association*, 64(1969), 923–937.